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A lower bound of the expected maximum number of edge-disjoint s–t paths on probabilistic graphs

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Abstract

For a probabilistic graph $(G = (V, E, s, t), p)$, where G is an undirected graph with specified source vertex s and sink vertex t ($s \neq t$) in which each edge has independent failure probability and each vertex is assumed to be failure-free, and $p = (p(e_1), \ldots, p(e_{|E|}))$ is a vector consisting of failure probabilities $p(e_i)$'s of all edges e_i's in E, we consider the problem of computing the expected maximum number $\Gamma_{(G,p)}$ of edge-disjoint s–t paths. It is known that this computing problem is NP-hard even if G is restricted to several classes like planar graphs, s–t out-in bitrees and s–t complete multi-stage graphs. In this paper, for a probabilistic graph $(G = (V, E, s, t), p)$, we propose a lower bound of $\Gamma_{(G,p)}$ and show the necessary and sufficient conditions by which the lower bound coincides with $\Gamma_{(G,p)}$. Furthermore, we also give a method of computing the lower bound of $\Gamma_{(G,p)}$ for a probabilistic graph $(G = (V, E, s, t), p)$.

1. Introduction

We consider a probabilistic graph $(G = (V, E, s, t), p)$, and G is an undirected graph with specified source vertex s and sink vertex t ($s \neq t$) in which each edge has independent failure probability and each vertex is assumed to be failure-free, and $p = (p(e_1), \ldots, p(e_{|E|}))$ is a vector consisting of failure probabilities $p(e_i)$'s of all edges e_i's in E. The expected maximum number $\Gamma_{(G,p)}$ of edge-disjoint s–t paths (namely, s–t paths having no edge in common) in a probabilistic graph (G, p) is useful for network reliability analysis. Note that the problem of computing s,t-connectedness [1,4], namely, probability that there exists at least one operative s–t path, is a special case of computing $\Gamma_{(G,p)}$ in a probabilistic graph (G, p).

However, it is known that the problem of computing $\Gamma_{(G,p)}$ in a probabilistic graph (G, p) is NP-hard, even if G is restricted to several classes, e.g., planar graphs, s–t out-in

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bitrees and s–t complete multi-stage graphs [2, 3]. Thus, for estimating $\Gamma_{(G,p)}$, it is interesting for us to find its lower bound in a probabilistic graph (G, p).

In this paper, we define a lower bound of $\Gamma_{(G,p)}$ using an s–t path number function of G for a probabilistic graph (G, p), and give the necessary and sufficient conditions by which this lower bound coincides with $\Gamma_{(G,p)}$ and a method of computing this lower bound. This paper is organized as follows:

Graph theoretic terminologies used throughout this paper are described in Section 2. A lower bound of $\Gamma_{(G,p)}$ in a probabilistic graph (G, p) is defined in Section 3. Section 4 shows the necessary and sufficient conditions by which this lower bound coincides with $\Gamma_{(G,p)}$. In Section 5, we suggest a method of computing the lower bound, and show that this computing method does not seem to efficiently compute this lower bound in general.

2. Preliminaries

2.1. Graph theoretic terminologies

A two-terminal undirected graph $G = (V, E, s, t)$ consists of a finite vertex set V and a finite undirected edge (unordered pair of vertices) set E, where s and t, called *source* and *sink*, respectively, are two specified distinct vertices of V. For an edge (u, v), the two vertices u and v are said to be end vertices of (u, v), and edge (u, v) is said to be incident to vertex u and vertex v.

In $G = (V, E, s, t)$, an *u–v path* π of length k from vertex u to vertex v is an alternating sequence of vertices $v_i \in V$ $(0 \leqslant i \leqslant k)$ and edges $(v_{i-1}, v_i) \in E$ $(1 \leqslant i \leqslant k)$,

$$\pi\colon\ v_0(=u), (v_0, v_1), v_1, \ldots, (v_{k-1}, v_k), v_k(=v),$$

where vertices v_i's $(0 \leqslant i \leqslant k)$ are distinct, i.e., a path denotes a simple path throughout this paper. For short, we also denote an u–v path by

$$\pi\colon\ v_0(=u), v_1, \ldots, v_{k-1}, v_k(=v).$$

The vertices $v_1, \ldots, v_{k-1}$ are called internal vertices of π and the vertices $v_0(=u)$, $v_k(=v)$ are called end vertices of π. Let $V(\pi), E(\pi)$ denote the set of all vertices and the set of all edges on an u–v path π, respectively. The set of all u–v paths in G is denoted by $P_{uv}(G)$. Paths $\pi_1, \ldots, \pi_r$ are called *internal vertex-disjoint paths* if they have no vertex in common except their end vertices. s–t paths $\pi_1, \ldots, \pi_r$ are called *edge-disjoint s–t paths* if any two of them have no edge in common, and the maximum number of edge-disjoint s–t paths in G is denoted by $\lambda_{st}(G)$.

A graph $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E, s, t)$, if $V_1 \subseteq V$ and $E_1 \subseteq E$. If G_1 is a subgraph of G, other than G itself, then G_1 is a proper subgraph of G. For a subset $E' \subseteq E$, the subgraph derived from G by deleting all edges of E' is denoted by $G - E'(=(V, E - E', s, t))$. A subset $E'(\subseteq E)$ is called an *s–t edge-cutset* if $G - E'$ has no s–t path. An s–t path π is said to be an *s–t edge-cut-path* if $E(\pi)$ is an s–t

edge-cutset. An s–t edge-cutset with the minimum cardinality among s–t edge-cutsets of G is said to be *minimum*. By well-known Menger's theorem [5], $\lambda_{st}(G)$ is equal to the cardinality of a minimum s–t edge-cutset for any $G = (V, E, s, t)$.

2.2. Probabilistic graph

A probabilistic graph, denoted by $(G = (V, E, s, t), p)$, or (G, p), for short, is defined as follows:

(i) In $G = (V, E, s, t)$, each edge e of E is in either of the following two states: failed or operative (not failed), having known independent failure probability $p(e)$, $0 \leqslant p(e) \leqslant 1$ (or operative probability $q(e) = 1 - p(e)$), and each vertex is assumed to be failure-free.

(ii) p is a vector consisting of all edge failure probabilities $p(e)$'s in E.

For a probabilistic graph $(G = (V, E, s, t), p)$, let a subgraph $G - U(\subseteq E)$ correspond to an event $\mathscr{E}_U$ that all edges of U are failed and all edges of $E - U$ are operative. Clearly, the probability $\rho(G - U)$ of arising a subgraph $G - U(\subseteq E)$ is computed by the following formula:

$$\rho(G - U) = \prod_{e \in U} p(e) \prod_{e \in E - U} q(e)(=1 - p(e)).$$

Furthermore, $\sum_{U \subseteq E} \rho(G - U) = 1$.

Now, we define the *expected maximum number* $\Gamma_{(G,p)}$ of edge-disjoint s–t paths in a probabilistic graph $(G = (V, E, s, t), p)$ as follows:

$$\Gamma_{(G,p)} \equiv \sum_{U \subseteq E} \lambda_{st}(G - U)\rho(G - U). \tag{1}$$

It is known that the problem of computing $\Gamma_{(G,p)}$ for a probabilistic graph (G, p) is NP-hard, even if G is restricted to several special classes like planar graphs, s–t out–in bitrees and s–t multi-stage complete graphs, etc. [2, 3]. Thus, it is interesting for us to consider a lower bound of $\Gamma_{(G,p)}$ for estimating it.

3. A lower bound of $\Gamma_{(G,p)}$

We define a lower bound of the expected maximum number of edge-disjoint s–t paths in a probabilistic graph.

An *s–t path number function* f of $G = (V, E, s, t)$ is a one-to-one integral function $f : P_{st}(G) \mapsto \{1, \ldots, |P_{st(G)}|\}$. The s–t path π with $f(\pi) = k$ is said to be the *s–t path of number k*, and denoted by π_k. The s–t path with the minimum number in $G - E'(\subseteq E)$ with respect to f is denoted by $\pi_{m(G-E',f)}$.

First, we give the following procedure **FEDP** to find a set of edge-disjoint s–t paths for $G = (V, E, s, t)$.

Procedure FEDP

Input: $G = (V, E, s, t)$ and an s–t path number function f of G.
Output: A set of edge-disjoint s–t paths $FEDP(G, f)$.
BEGIN
  $G' := G$; $FEDP(G, f) := \emptyset$;
  WHILE $P_{st}(G') \neq \emptyset$ DO
    BEGIN
      Find $\pi_{m(G', f)}$ from $P_{st}(G')$;
      $FEDP(G, f) := FEDP(G, f) \cup \{\pi_{m(G', f)}\}$;
      $G' := G' - E(\pi_{m(G', f)})$
    END;
  Output $FEDP(G, f)$
  END.

Note that s–t path $\pi_{m(G', f)}$ is arbitrarily selected from $P_{st}(G')$. It is clear that $FEDP(G, f)$ obtained by **FEDP** is a set of edge-disjoint s–t paths in $G = (V, E, s, t)$. Namely, the following formula holds.

$$|FEDP(G, f)| \leqslant \lambda_{st}(G) \quad \text{for any } G, f. \tag{2}$$

For a probabilistic graph $(G = (V, E, s, t), p)$ and an s–t path number function f of G, we now define the value $\underline{\Gamma}_{(G, f, p)}$ as follows:

$$\underline{\Gamma}_{(G, f, p)} \equiv \sum_{U \subseteq E} |FEDP(G - U, f)| \rho(G - U). \tag{3}$$

By formulas (1)–(3), $\underline{\Gamma}_{(G, f, p)}$ is a lower bound of $\Gamma_{(G, p)}$, namely, the following formula holds:

$$\underline{\Gamma}_{(G, f, p)} \leqslant \Gamma_{(G, p)} \quad \text{for any } G, f, p.$$

4. Necessary and sufficient conditions

In this section, we give the necessary and sufficient conditions by which $\underline{\Gamma}_{(G, f, p)}$ côincides with $\Gamma_{(G, p)}$ in a probabilistic graph (G, p).

4.1. Necessary and sufficient conditions of an s–t path number function

By formulas (1)–(3), the following Theorem 4.1 immediately holds.

Theorem 4.1. *For a probabilistic graph $(G = (V, E, s, t), p)$, $\underline{\Gamma}_{(G, f, p)} = \Gamma_{(G, p)}$ iff G has an s–t path number function f satisfying the following formula*:

$$|FEDP(G - U, f)| = \lambda_{st}(G - U) \quad \textit{for any } U \subseteq E. \tag{4}$$

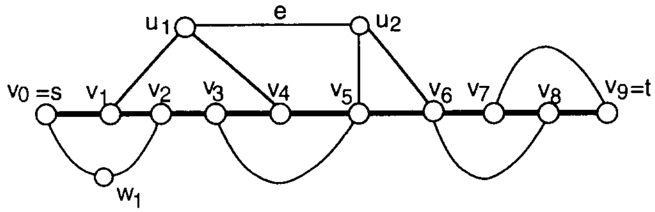

Fig. 1. A π-edge-cut s–t 2-edge-connected graph.

Definition 4.1. An s–t path number function f of $G = (V, E, s, t)$ is said to be *exact* if f satisfies formula (4).

$G = (V, E, s, t)$ is said to be s–t *k-edge-connected* if $\lambda_{st}(G) \geqslant k$. $G = (V, E, s, t)$ is said to be π-edge-cut if π is an s–t edge-cut-path in G. $G = (V, E, s, t)$ is said to be *π-edge-cut s–t 2-edge-connected* if π is an s–t edge-cut-path of G and G is s–t 2-edge-connected. A π-edge-cut s–t 2-edge-connected graph $G = (V, E, s, t)$ is *minimal*, if $G - \{e\}$ for any $e \in E - E(\pi)$ is not π-edge-cut s–t 2-edge-connected. For example, the graph shown in Fig. 1 is π-edge-cut s–t 2-edge-connected, where π: $v_0(=s), v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9(=t)$. But it is not minimal as $G - \{e = (u_1, u_2)\}$ is π-edge-cut s–t 2-edge-connected. Furthermore, the set of all minimal π-edge-cut s–t 2-edge-connected subgraphs of an s–t path π in G is denoted by $\mathscr{W}(G, \pi)$. For example, in the graph G given in Fig. 1, $\mathscr{W}(G, \pi) = \{G - \{e\},\ G - \{(u_1, v_4), (u_2, v_5), (v_3, v_5)\}\}$. Clearly, the following lemma holds.

Lemma 4.1. *For $G = (V, E, s, t)$, if $\lambda_{st}(G) \geqslant 2$ holds and an s–t path π of G is an s–t edge-cut-path, then $\mathscr{W}(G, \pi) \neq \emptyset$.*

Lemma 4.2. *If there exists an s–t path π satisfying $\mathscr{W}(G, \pi) = \emptyset$ in $G = (V, E, s, t)$, then the following formula holds*:

$$\lambda_{st}(G - E(\pi)) = \lambda_{st}(G) - 1.$$

Proof. Clearly, $\lambda_{st}(G - E(\pi)) \leqslant \lambda_{st}(G) - 1$. Assume that $\lambda_{st}(G - E(\pi)) < \lambda_{st}(G) - 1$. By this assumption, there exists a minimum s–t edge-cutset E^* in $G - E(\pi)$ that satisfies $|E^*| \leqslant \lambda_{st}(G) - 2$ by Menger's theorem [5]. Consider subgraph $G - E^*$, and it is clear that all s–t paths in $G - E^*$ share at least one edge of $E(\pi)$, i.e., π is an s–t edge-cut-path of $G - E^*$. Furthermore, let E' be a minimum s–t edge-cutset of $G - E^*$. As $E' \cup E^*$ is an s–t edge-cutset of G, $|E' \cup E^*| = |E'| + |E^*| \geqslant \lambda_{st}(G)$. By $|E^*| \leqslant \lambda_{st}(G) - 2$, we obtain $|E'| = \lambda_{st}(G - E^*) \geqslant 2$, contradicting the fact that $\mathscr{W}(G, \pi) \neq \emptyset$ by Lemma 4.1. $\square$

We now prove the following theorem.

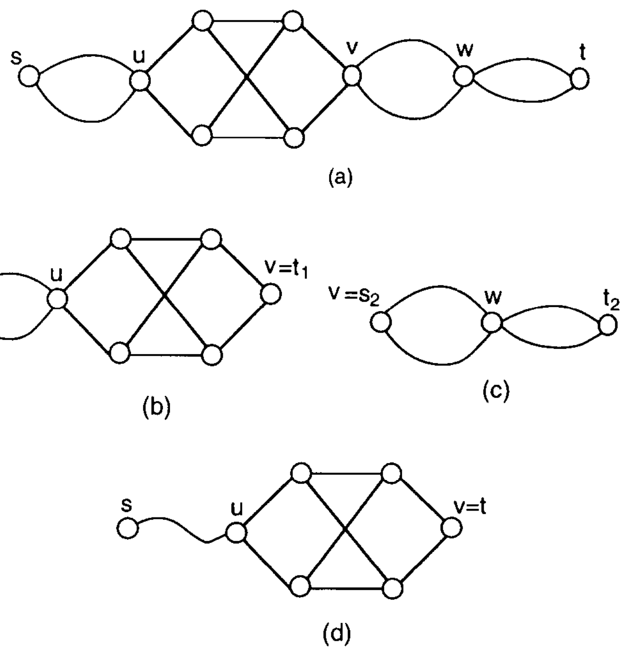

Fig. 2. An illustration of separation of G at an s–t 2-edge-connected articulation vertex.

Theorem 4.2. *In $G = (V, E, s, t)$, an s–t path number function f of G is exact iff for any $U \subseteq E$ with $P_{st}(G - U) \neq \emptyset$, $\mathscr{W}(G - U, \pi_{m(G-U,f)}) = \emptyset$.*

Proof. Necessity: Assume that an s–t path number function f of G is exact and that for some $U \subseteq E$ with $P_{st}(G - U) \neq \emptyset$, $\mathscr{W}(G - U, \pi_{m(G-U,f)}) \neq \emptyset$. By $\mathscr{W}(G - U, \pi_{m(G-U,f)}) \neq \emptyset$, $G - U$ has a subgraph $G' \in \mathscr{W}(G - U, \pi_{m(G-U,f)})$. $\lambda_{st}(G') = 2$ by the definition of $\mathscr{W}(G - U, \pi_{m(G-U,f)})$. As $\pi_{m(G-U,f)}$ is the s–t path with the minimum number of G' and an s–t edge-cut-path of G', we have $FEDP(G', f) = \{\pi_{m(G-U,f)}\}$ by **FEDP**. Hence, $|FEDP(G', f)|(=1) < \lambda_{st}(G')(=2)$, contradicting the fact that f is exact.

Sufficiency: Assume that for any $U \subseteq E$ with $P_{st}(G - U) \neq \emptyset$, $\mathscr{W}(G - U, \pi_{m(G-U,f)}) = \emptyset$. Then it is easy to prove that for any $U \subseteq E$, $|FEDP(G - U, f)| = \lambda_{st}(G - U)$ by iteratively applying Lemma 4.2. □

Definition 4.2 (*Prohibitive s–t path subset*). In $G = (V, E, s, t)$, a subset P of the s–t path set $P_{st}(G)$ is called a *prohibitive s–t path subset* if, for each s–t path π of P, there is a π-edge-cut s–t 2-edge-connected subgraph $G_\pi \in \mathscr{W}(G, \pi)$ such that s–t path set $P_{st}(G_\pi)$ is contained in P, i.e., $P_{st}(G_\pi) \subseteq P$.

For example, each of the graphs shown in Fig. 2(a) and (b) has a prohibitive s–t path subset, namely, s–t path set itself, and each of the graphs shown in Fig. 2(c) and (d) has no prohibitive s–t path subset. Furthermore, we introduce the following procedure **TEST** to tell if $G = (V, E, s, t)$ contains a prohibitive s–t path subset or an exact s–t path number function f of G exists.

Procedure TEST
Input: $G = (V, E, s, t)$.
Output: Either an s–t path number function f of G or a subset P of $P_{st}(G)$.
BEGIN
  $P := P_{st}(G)$; $i := 1$; $Q := \{\pi \in P_{st}(G) \mid \mathscr{W}(G, \pi) = \emptyset\}$;
  WHILE $Q \neq \emptyset$ DO
    BEGIN
      $P := P - Q$;
      REPEAT
        Select an s–t path π from Q;
        $f(\pi) := i$; $i := i + 1$; $Q := Q - \{\pi\}$
      UNTIL $Q = \emptyset$;
      $Q := \{\pi \in P \mid P_{st}(G_\pi) \nsubseteq P$ *for all* $G_\pi \in \mathscr{W}(G, \pi)\}$
    END;
    IF $P = \emptyset$ THEN output f ELSE output P
END.

Clearly, the following Lemma 4.3 holds by Definitions 4.1 and 4.2.

Lemma 4.3. *If* **TEST** *outputs an s–t path number function f of G, then f is exact, when $G = (V, E, s, t)$ is input. If* **TEST** *outputs a subset P of $P_{st}(G)$, then P is a prohibitive s–t path subset, when $G = (V, E, s, t)$ is input.*

In $G = (V, E, s, t)$, if there is a prohibitive s–t path subset $P(\subseteq P_{st}(G))$, then there does not exist any exact s–t path number function f. Otherwise, if G has an exact s–t path number function f, and suppose π_m is the s–t path of the minimum number with respect to f among P. By Definition 4.2, there is $G_{\pi_m} \in \mathscr{W}(G, \pi_m)$ in G that satisfies $P_{st}(G_{\pi_m}) \subseteq P$. Thus, π_m is also the s–t path of the minimum number with respect to f in G_{π_m}. Therefore, by **FEDP**, $FEDP(G_{\pi_m}, f) = 1 < \lambda_{st}(G_{\pi_m}) = 2$. This leads to a contradiction that f is an exact s–t path number function of G. Hence, by Theorem 4.2 and Lemma 4.3, the following theorem holds.

Theorem 4.3. *$G = (V, E, s, t)$ has an exact s–t path number function iff it contains no prohibitive s–t path subset.*

4.2. Characterization of graph having a prohibitive s–t path subset

A graph is connected if there is a path connecting each pair of vertices and otherwise disconnected. A connected component of a graph is a maximal connected

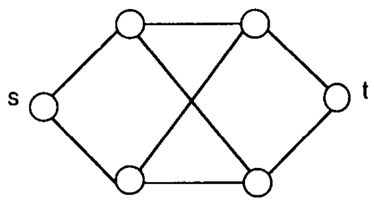

Fig. 3. A prohibitive graph.

subgraph, which is simply called a component. If there exist distinct vertices u, v and w in a graph such that all the paths connecting u and w contain v as an internal vertex, then v is an *articulation vertex*. A two-terminal connected graph is said to be s, t *non-separable* if its subgraph obtained by removing s, t is connected. In the following discussion, we assume that $G = (V, E, s, t)$ is an s, t non-separable two-terminal connected graph, unless otherwise specified.

Definition 4.3 (*s–t 2-edge-connected articulation vertex*). A vertex v of $G = (V, E, s, t)$ is said to be an *s–t 2-edge-connected articulation vertex*, if v is an s–t articulation vertex of G and there exist both two edge-disjoint s–v paths and two edge-disjoint v–t paths in G.

For example, in the graph illustrated in Fig. 2(a), vertices u, v, w are s–t 2-edge-connected articulation vertices.

Definition 4.4 (*Separation of G at an s–t 2-edge-connected articulation vertex*). Assume that $G = (V, E, s, t)$ has an s–t 2-edge-connected articulation vertex v. The following sequence of operations is said to be *separation of G at an s–t 2-edge-connected articulation vertex v*.

(i) The two components C_1 and C_2 are obtained by removing v from G.
(ii) v is connected to C_1 (or C_2) with all edges (u, v)'s of G having one end vertex u in C_1 (or C_2).
(iii) Note that C_1 contains either of s, t. If C_1 contains s (or t) then let s (or t) be s_1 (or t_1) and let v be t_1 (or s_1). s_2 and t_2 are similarly defined for C_2.

For example, the two graphs illustrated in Fig. 2(b), (c) are obtained by the separation of the graph given in Fig. 2(a) at an s–t 2-edge-connected articulation vertex v.

Definition 4.5 (*Prohibitive graph*). $G = (V, E, s, t)$ is said to be a *prohibitive graph*, if G, or one of the graphs derived from G by separations of G at all s–t 2-edge-connected articulation vertices in G is homeomorphic to the graph shown in Fig. 3.

The two graphs illustrated in Fig. 2(a), (b) are both prohibitive graphs. But the graph given in Fig. 2(d) (although it contains a subgraph homeomorphic to the graph shown in Fig. 3) is not a prohibitive graph as the vertex u is not its s–t 2-edge-connected articulation vertex and it is not homeomorphic to the graph shown in Fig. 3. It is easy to verify that for a prohibitive graph $G = (V, E, s, t)$, $P_{st}(G)$ is a prohibitive s–t path subset. Thus, we immediately obtain the following lemma.

Lemma 4.4. *If $G = (V, E, s, t)$ contains a prohibitive graph as its subgraph, then it has a prohibitive s–t path subset.*

Furthermore, we show that if $G = (V, E, s, t)$ has a prohibitive s–t path subset, then it contains a prohibitive graph as its subgraph. For our aim, we need more definitions.

Definition 4.6 (*Attachment vertex* [6, 7]). An *attachment vertex* of a subgraph G_1 in G is a vertex of G_1 incident in G with some edge not belonging to G_1.

Definition 4.7 (*Bridges* [6, 7]). Let J be a fixed subgraph of G. A subgraph G_1 of G is said to be *J-detached* in G if all its attachment vertices are in J. We define a *bridge* of J in G as any subgraph B that satisfies the following three conditions:

(i) B is not a subgraph of J.
(ii) B is J-detached in G.
(iii) No proper subgraph of B satisfies both (i) and (ii).

Definition 4.8 (*Degenerate and proper bridges. Nucleus of a bridge* [6, 7]). An edge $e = (u, v)$ of G not belonging to J but having both end vertices in J is referred to as a *degenerate bridge.*

Let G^- be the graph derived from G by deleting the vertices of J and all edges incident to them. Let C be any component of G^-. Let B be the subgraph of G obtained from C by adjoining to it each edge of G having one end vertex in C and the other end vertex in J and adjoining also the end vertices in J of all such edges. The subgraph B satisfies the conditions (i)–(iii) in Definition 4.7 and is a bridge. Such a bridge is called *proper*. The component C of G^- is the *nucleus* of B.

For the graph shown in Fig. 4, let J be an s–t path π: $v_0(=s), v_1, v_2, v_3, v_4, v_5, v_6(=t)$, then all vertices on π other than v_4 are all attachment vertices of π in the graph. B_1, B_2 and B_3 are proper bridges of π in the graph and B_4 is a degenerate bridge of π in the graph. By Definitions 4.6 and 4.7, the following lemma obviously holds.

Lemma 4.5. *Let π be an s–t path in $G = (V, E, s, t)$. If there is a proper bridge B of π in G, then any two vertices, u, v in B are connected by a path consisting of edges and vertices only in the nucleus of B.*

Let γ: $v_0, v_1, \ldots, v_{k-1}, v_k$ be a path from v_0 to v_k in a graph. A subsequence $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ of γ for $0 \leqslant i < j \leqslant k$ is called a subpath of γ, and denoted by

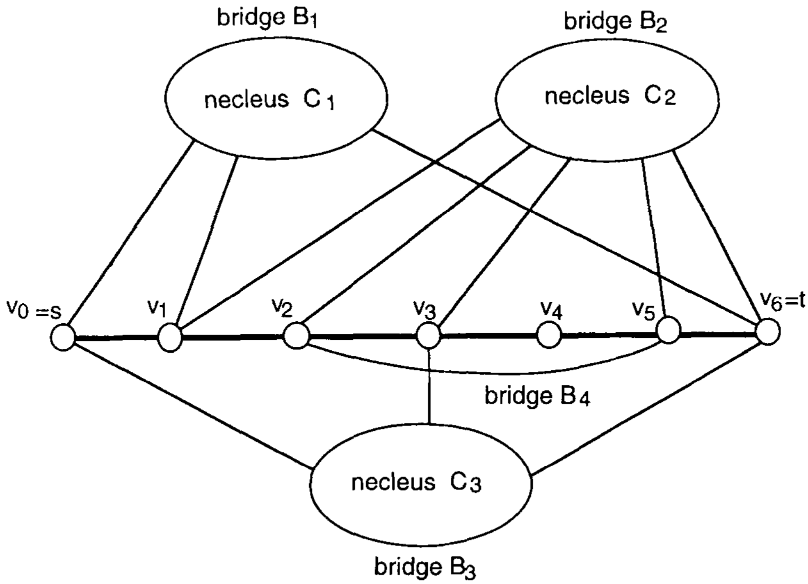

Fig. 4. An illustration of attachment vertices, bridges and nuclei.

$\gamma[v_i, v_j]$. If γ': $v_0, v_1, \ldots, v_{i-1}, v_i$ and γ'': $v_i, v_{i+1}, \ldots, v_{k-1}, v_k$ are two internal vertex-disjoint paths, then the *concatenation* $\gamma' * \gamma''$ of γ' and γ'' is $v_0, v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{k-1}, v_k$.

Definition 4.9 (*Path avoiding s–t path π*). Let π be an s–t path in $G = (V, E, s, t)$. For two vertices v_i, v_j in $V(\pi)$, a path between v_i and v_j consisting of edges not in $E(\pi)$ and vertices not in $V(\pi)$ except v_i, v_j is said to be *avoiding* π.

For example, the path v_1, u_1, u_2, v_5 is avoiding the s–t path π in the graph illustrated in Fig. 1.

Definition 4.10 (*Order relation with respect to an s–t path*). Let π: $v_0(=s)$, $v_1, \ldots, v_{k-1}, v_k(=t)$ be an s–t path in $G = (V, E, s, t)$. We define an *order relation* $<_\pi$ over $V(\pi)$ with respect to π as follows: For any v_i, v_j $(0 \leqslant i, j \leqslant k)$, $v_i <_\pi v_j$ iff $i < j$. If $v_i <_\pi v_j$, v_i (v_j) is said to be to the left (right) of v_j (v_i).

Definition 4.11 (*Intersection vertex of two paths π, α*). Let π, α be two paths in $G = (V, E, s, t)$. A vertex v in G is called an *intersection vertex* of π, α if π, α arrive at v or leave from v by two distinct edges. The set of all intersection vertices of π, α is denoted by $V_{\pi\alpha}(\subseteq V(\pi) \cap V(\alpha))$.

In the graph given in Fig. 1, for two s–t paths π and α: $v_0(=s)$, v_1, u_1, u_2, v_6, v_7, $v_9(=t)$, we have $V_{\pi\alpha} = \{v_1, v_6, v_7, v_9\}$.

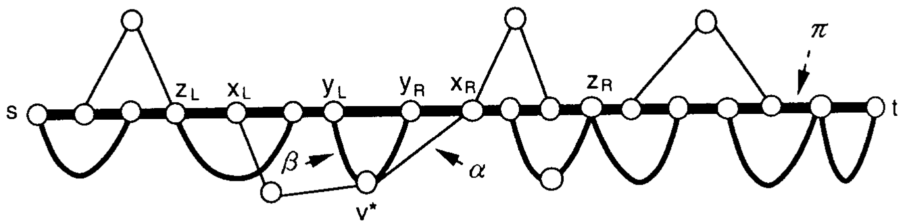

Fig. 5. An illustration of the proof of Lemma 4.6(ii).

Lemma 4.6. *Suppose that $G = (V, E, s, t)$ has an s–t path π: $v_0(= s), v_1, \ldots, v_{k-1}, v_k(= t)$ satisfying $\mathscr{W}(G, \pi) \neq \emptyset$.* Let *$G_\pi \in \mathscr{W}(G, \pi)$ be a minimal π-edge-cut s–t 2-edge-connected subgraph of G, and let α, β be two edge-disjoint s–t paths in G_π. Let $V_{\pi\alpha} = \{x_1, x_2, \ldots, x_p\}$ be the set of all intersection vertices of π, α, where $x_1 <_\pi x_2 <_\pi \cdots <_\pi x_p$. Let $V_{\pi\beta} = \{y_1, y_2, \ldots, y_q\}$ be the set of all intersection vertices of π, β, where $y_1 <_\pi y_2 <_\pi \cdots <_\pi y_q$. Let $V_{\pi\alpha\beta} = \{z_1, z_2, \ldots, z_r\}$ be the set of all vertices which π, α, β have in common, where $z_1 <_\pi z_2 <_\pi \cdots <_\pi z_r$. Then*

(i) *For a subpath $\alpha[x_i, x_{i+1}]$ of α avoiding π, if there is no y_j in $V_{\pi\beta}$ satisfying $x_i <_\pi y_j <_\pi x_{i+1}$, then $\pi[x_i, x_{i+1}]$ is a subpath of β, namely, $\pi[x_i, x_{i+1}] = \beta[x_i, x_{i+1}]$. (For a subpath $\beta[y_j, y_{j+1}]$ of β avoiding π, if there is no x_i in $V_{\pi\alpha}$ satisfying $y_j <_\pi x_i <_\pi y_{j+1}$, then $\pi[x_i, x_{i+1}]$ is a subpath of α, namely, $\pi[y_j, y_{j+1}] = \alpha[x_i, x_{i+1}]$.)*

(ii) *For any z_k, z_{k+1} $(1 \leqslant k \leqslant r - 1)$, the two subpaths $\alpha[z_k, z_{k+1}]$, $\beta[z_k, z_{k+1}]$ are vertex-disjoint other than z_k, z_{k+1}.*

Proof. (i) Otherwise, assume that $\pi[x_i, x_{i+1}]$ is not a subpath of β, i.e. there is an edge in $\pi[x_i, x_{i+1}]$ but not in β. As there is no y_j in $V_{\pi\beta}$ satisfying $x_i <_\pi y_j <_\pi x_{i+1}$, $\pi[x_i, x_{i+1}]$ is edge-disjoint with β by this assumption and the definition of $V_{\pi\beta}$. Thus, β and $\alpha[s, x_i] * \pi[x_i, x_{i+1}] * \alpha[x_{i+1}, t]$ are two edge-disjoint s–t paths in $G_\pi - E(\alpha[x_i, x_{i+1}])$. Note that π is an s–t edge-cut-path in $G_\pi - E(\alpha[x_i, x_{i+1}])$. Hence, $G_\pi - E(\alpha[x_i, x_{i+1}])$ is a π-edge-cut s–t 2-edge-connected subgraph in G. This contradicts the fact that G_π is a minimal π-edge-cut s–t 2-edge-connected subgraph of G.

(ii) Assume that the two subpaths $\alpha[z_k, z_{k+1}]$ of α and $\beta[z_k, z_{k+1}]$ of β have a vertex $v^*(\neq z_k, z_{k+1})$ in common. Clearly, v^* is not on $\pi[z_k, z_{k+1}]$. Let $x_L \in V_{\pi\alpha}$ be the rightmost vertex on α to the left of v^* and let $x_R \in V_{\pi\alpha}$ be the leftmost vertex on α to the right of v^*. Likewise, let $y_L \in V_{\pi\beta}$ be the rightmost vertex on β to the left of v^* and let $y_R \in V_{\pi\beta}$ be the leftmost vertex on β to the right of v^*. (See Fig. 5.) Note that x_L, x_R are distinct from y_L, y_R, respectively, unless $x_L = y_L = z_L$ and $x_R = y_R = z_R$.

In the case of $x_L <_\pi y_L <_\pi y_R <_\pi x_R$, the two s–t paths $\alpha, \beta' = \beta[s, y_L] * [y_L, y_R] * \beta[y_R, t]$ are edge-disjoint as $\pi[y_L, y_R]$ is avoiding β. Furthermore, let G' be the subgraph obtained from G_π be deleting all edges on $\beta[y_L, v^*]$ and $\beta[v^*, y_R]$. By Lemma 4.1, G' is π-edge-cut s–t 2-edge-connected, as π is an s–t edge-cut-path and G'

is s–t 2-edge-connected. This contradicts the fact that G_π is a minimal π-edge-cut s–t 2-edge-connected subgraph of G.

Discussions similar to the case of $x_L <_\pi y_L <_\pi y_R <_\pi x_R$ prove other cases. □

Definition 4.12 (*Interlacing subpaths*). Suppose that $G = (V, E, s, t)$ has an s–t path π: $v_0(=s), v_1, \ldots, v_{k-1}, v_k(=t)$ satisfying $\mathscr{W}(G, \pi) \neq \emptyset$. Let $G_\pi \in \mathscr{W}(G, \pi)$ be a minimal π-edge-cut s–t 2-edge-connected subgraph of G. Let α, β be two edge-disjoint s–t paths in G_π. Let $V_{\pi\alpha} = \{x_1, x_2, \ldots, x_p\}$ be the set of all intersection vertices of π, α, where $x_1 <_\pi x_2 <_\pi \cdots <_\pi x_p$. Let $V_{\pi\beta} = \{y_1, y_2, \ldots, y_q\}$ be the set of all intersection vertices of π, β, where $y_1 <_\pi y_2 <_\pi \cdots <_\pi y_q$. Let $V_{\pi\alpha\beta} = \{z_1, z_2, \ldots, z_r\}$ be the set of all vertices which π, α, β have in common, where $z_1 <_\pi z_2 <_\pi \cdots <_\pi z_r$. Subpaths $\alpha[x_i, x_{i+1}]$ of α avoiding π and $\beta[y_j, y_{j+1}]$ of β avoiding π, where either $x_i <_\pi y_j$ or $y_j <_\pi x_i$, are said to be *interlacing subpaths*, if the subpath $\pi[x_i, y_{j+1}]$ ($\pi[y_j, x_{i+1}]$) contains no vertex of $V_{\pi\alpha\beta}$ when $x_i <_\pi y_j$ ($y_j <_\pi x_i$).

In the graph shown in Fig. 1, for two edge-disjoint s–t paths; α: $v_0(=s), v_1, u_1, v_4, v_5, u_2, v_6, v_7, v_9(=t)$, β: $v_0(=s), w_1, v_2, v_3, v_5, v_6, v_8, v_9(=t)$, we have $V_{\pi\alpha} = \{v_1, v_4, v_5, v_6, v_7, v_9\}$, $V_{\pi\beta} = \{v_0, v_2, v_3, v_5, v_6, v_8\}$, $V_{\pi\alpha\beta} = \{v_0, v_5, v_6, v_9\}$. $\alpha[v_1, v_4]$, $\beta[v_0, v_2]$ are interlacing subpaths, and $\alpha[v_7, v_9]$, $\beta[v_6, v_8]$ are also interlacing paths. But $\alpha[v_1, v_4]$, $\beta[v_6, v_8]$ are not interlacing subpaths as $v_5, v_6 \in V_{\pi\alpha\beta}$ are on $\pi[v_0, v_8]$. By Definition 4.12 and Lemma 4.6, we have the following lemma.

Lemma 4.7. *Suppose that $G = (V, E, s, t)$ has an s–t path π: $v_0(=s), v_1, \ldots, v_{k-1}, v_k(=t)$ satisfying $\mathscr{W}(G, \pi) \neq \emptyset$. Let $G_\pi \in \mathscr{W}(G, \pi)$ be a minimal π-edge-cut s–t 2-edge-connected subgraph of G. Let α, β be two edge-disjoint s–t paths in G_π. Let $V_{\pi\alpha} = \{x_1, x_2, \ldots, x_p\}$, $V_{\pi\beta} = \{y_1, y_2, \ldots, y_q\}$ and $V_{\pi\alpha\beta} = \{z_1, z_2, \ldots, z_r\}$ be as in Definition* 4.12. *If there are two interlacing subpaths $\alpha[x_i, x_{i+1}]$, $\beta[y_j, y_{j+1}]$ of α, β, respectively, where $x_i <_\pi y_j$, then there is a subpath $\beta[y_{j'}, y_{j'+1}]$ of β satisfying $x_i <_\pi y_{j'} <_\pi x_{i+1} <_\pi y_{j'+1}$ such that $\alpha[x_i, x_{i+1}]$, $\beta[y_{j'}, y_{j'+1}]$ are interlacing subpaths.*

In order to show that if $G = (V, E, s, t)$ has a prohibitive s–t path subset $P(\subseteq P_{st}(G))$, then it must contain a prohibitive graph as its subgraph, we now prove the following two lemmas.

Lemma 4.8. *If $G = (V, E, s, t)$ has a prohibitive s–t path subset $P(\subseteq P_{st}(G))$, then P contains an s–t path π such that some proper bridge B of π in G has two interlacing subpaths $\alpha[x_i, x_{i+1}]$ of α and $\beta[y_j, y_{j+1}]$ of β with respect to π in G_π, where G_π is a minimal π-edge-cut s–t 2-edge-connected subgraph of G, and α, β are two edge-disjoint s–t paths in G_π.*

Proof. Assume that P contains no s–t path π such that some proper bridge B of π in G has interlacing subpaths $\alpha[x_i, x_{i+1}]$ of α and $\beta[y_j, y_{j+1}]$ of β with respect to π in G_π, where G_π is a minimal π-edge-cut s–t 2-edge-connected subgraph of G, and α, β are two edge-disjoint s–t paths in G_π. (We call this *Assumption* 1.)

By Definition 4.8, there is an s–t path π_0 of P having a proper bridge of π_0 in G. Let B_0 be a proper bridge of π_0 in G. Let P_0 be the s–t path set derived from P by deleting all s–t paths of P having at least one vertex in the nucleus of B_0. If there is an s–t path π' in P_0 satisfying the following condition:

Condition A: *At least one of attachment vertices of B_0 of π_0 in G is not contained in the s–t path π'.*

Then we modify π_0, B_0, P_0 as follows:

Let π' be an s–t path of P_0 satisfying *Condition A*.

Let B' be the bridge of π' in G containing the nucleus of bridge B_0.

Let P' be the s–t path set derived from P_0 by deleting all s–t paths of P_0 having at least one vertex in the nucleus of B'.

Let now π_0, B_0, P_0 be π', B', P', respectively.

We iteratively modify π_0, B_0 and P_0 until P_0 has no s–t path satisfying *Condition A*. Thus, for π_0, B_0 and P_0, we have the following facts:

Fact 1. *In each modification, the number of s–t path of P_0 decreases at least by one, as π' is deleted from P_0.*

Fact 2. *B_0 is a bridge of any s–t path of P_0 in G, as any s–t path of P_0 has no vertex in the nucleus of B_0 by modifying P_0 and has all attachment vertices of B_0 in G by Condition A.*

Fact 3. *For any $\pi \in P_0 \cup \{\pi_0\}$, B_0 does not contain any one of two interlacing subpaths $\alpha[x_i, x_{i+1}]$ of α and $\beta[y_j, y_{j+1}]$ of β, where α, β are two edge-disjoint s–t paths in $G_\pi \in \mathscr{W}(G, \pi)$ satisfying $P(G_\pi) \subseteq P$. Otherwise, assume that B_0 contains the subpath $\alpha[x_i, x_{i+1}]$ of α. By Lemma 4.7, we can assume that $\beta[y_j, y_{j+1}]$ of β satisfies either $y_j <_\pi x_i <_\pi y_{j+1}$ or $y_j <_\pi x_{i+1} <_\pi y_{j+1}$. Without loss of generality, we assume that $y_j <_\pi x_i <_\pi y_{j+1}$. By Fact 2, both x_i, x_{i+1} are attachment vertices of B_0 of π_0 in G. By Assumption 1, B_0 does not contain the subpath $\beta[y_j, y_{j+1}]$ of β. At least one of attachment vertices of B_0 of π_0 in G is not contained in s–t path $\pi' = \pi[s, y_j] * \beta[y_j, y_{j+1}] * \pi[y_{j+1}, t]$ as the attachment vertex x_i is not in $V(\pi')$. π' is in P as $\pi' \in P_{st}(G_\pi) \subseteq P$. Hence, by modifying P_0, π' is in P_0 satisfying Condition A, contradicting the fact that P_0 has no s–t path satisfying Condition A.*

Proof of Lemma 4.8 (Conclusion). Further, if $P_0 \neq \emptyset$, then if there is a proper bridge of π_0 in G different from B_0 then let $\pi = \pi_0$, B_1 be a proper bridge of π_0 in G different from B_0, $P_1 = P_0$, otherwise, let π be an s–t path of P_0, B_1 be a proper bridge of π_0 in G different from B_0, $P_1 = P_0 \cup \{\pi_0\} - \{\pi_1\}$. Similarly, we iteratively modify π_1, B_1, P_1 until P_1 has no s–t path satisfying *Condition A*. For π_1, B_1 and P_1, we also have the same facts as that for π_0, B_0 and P_0. Clearly, $P_1 \subset P_0$.

By *Fact* 1, we can stop this modification until $P_i = \emptyset$. As $\pi_i \in P$, there is $G_{\pi_i} \in \mathscr{W}(G, \pi_i)$ satisfying $P_{st}(G_{\pi_i}) \subseteq P$. Let α_i, β_i be two edge-disjoint s–t paths in G_{π_i} and let

$\alpha_i[x_{i'}, x_{i'+1}]$ of α_i and $\beta_i[y_{j'}, y_{j'+1}]$ of β_i be two interlacing subpaths. Any one of $\alpha_i[x_{i'}, x_{i'+1}]$ and $\beta_i[y_{j'}, y_{j'+1}]$ is not contained in one of bridges $B_0, B_1, \ldots, B_{i-1}$, otherwise contradicting *Fact* 3. By *Assumption* 1, B_i contains at most one of $\alpha_i[x_{i'}, x_{i'+1}]$ and $\beta_i[y_{j'}, y_{j'+1}]$. Without loss of generality, we assume that B_i does not contain $\alpha_i[x_{i'}, x_{i'+1}]$. Thus, s–t path $\pi' = \pi_i[s, x_{i'}] * \alpha_i[x_{i'}, x_{i'+1}] * \pi_i[x_{i'+1}, t]$ is in G_{π_i} and of P. As π_i does not have any vertex in the nucleus of any of bridges $B_0, B_1, \ldots, B_i$ by modifying $P_0, P_1, \ldots, P_i$, π' also has no vertex in the nucleus of any of bridges $B_0, B_1, \ldots, B_i$. Hence, π' also is in P_i by modifying P_i. This contradicts $P_i = \emptyset$. $\square$

Lemma 4.9. *Suppose that $G = (V, E, s, t)$ has an s–t path π satisfying $\mathscr{W}(G, \pi) \neq \emptyset$. Let α, β be two edge-disjoint s–t paths in $G_\pi \in \mathscr{W}(G, \pi)$. Let $V_{\pi\alpha} = \{x_1, x_2, \ldots, x_p\}$, $V_{\pi\beta} = \{y_1, y_2, \ldots, y_q\}$ and $V_{\pi\alpha\beta} = \{z_1, z_2, \ldots, z_r\}$ be defined as in Definition 4.12. If a bridge B of π in G contains interlacing subpaths $\alpha[x_i, x_{i+1}]$ of α and $\beta[y_j, y_{j+1}]$ of β in G_π with respect to π. Then G contains a prohibitive graph as its subgraph.*

Proof. By the known conditions given in this lemma, we construct a prohibitive graph as its subgraph.

By Lemma 4.5, there is a path π_{uv} between an internal vertex u on $\alpha[x_i, x_{i+1}]$ and an internal vertex v on $\beta[y_j, y_{j+1}]$ consisting of edges and vertices only in the nucleus of bridge B, i.e., π_{uv} is vertex-disjoint with π except u, v. Without loss of generality, assume that $x_i <_\pi y_j$. Let $z_L \in V_{\pi\alpha\beta}$ be the rightmost vertex to the left of x_i and let $z_R \in V_{\pi\alpha\beta}$ be the leftmost vertex to the right of y_{j+1}. Namely, $\pi[z_L, z_R]$ has no vertex in $V_{\pi\alpha\beta}$ as $\pi[x_i, y_{j+1}]$ has no vertex in $V_{\pi\alpha\beta}$. (See Fig. 6.) By Lemma 4.6(ii), the two subpaths $\alpha[z_L, z_R]$, $\beta[z_L, z_R]$ are vertex-disjoint other than z_L, z_R.

Without loss of generality, assume that path π_{uv}: $w_0(=u), w_1, \ldots, w_l(=v)$ has no vertex on both $\alpha[z_L, z_R]$ and $\beta[z_L, z_R]$ except w_0, w_l. Furthermore, let $w_L \in V(\pi_{uv})$ be the leftmost vertex shared by π_{uv} and the leftmost subpath $\gamma_L[x^L_L, x^L_{L+1}]$ among all subpaths of α, β (namely, $\alpha[x_i, x_{i+1}]$'s and $\beta[y_j, y_{j+1}]$'s) avoiding π, where either ($\gamma_L = \alpha$ and $\gamma'_L = \beta$) or ($\gamma_L = \beta$ and $\gamma'_L = \alpha$) holds. Let $w_R \in V(\pi_{uv})$ be the rightmost vertex shared by π_{uv} and the rightmost subpath $\gamma_R[x^R_R, x^R_{R+1}]$ among all subpaths of α, β (namely, $\alpha[x_i, x_{i+1}]$'s and $\beta[y_j, y_{j+1}]$'s) avoiding π, where either ($\gamma_R = \alpha$ and $\gamma'_R = \beta$) or ($\gamma_R = \beta$ and $\gamma'_R = \alpha$) holds. Let $z_{L^*} \in V_{\pi\alpha\beta}$ be the rightmost vertex to the left of x^L_L and let $z_{R^*} \in V_{\pi\alpha\beta}$ be the leftmost vertex to the right of x^R_{R+1}. Let $y_{w_L} \in V_{\pi\gamma'_L}$ be the leftmost vertex to the right of x^L_L on $V_{\pi\gamma'_L}$ and let $y_{w_R} \in V_{\pi\gamma'_R}$ be the rightmost vertex to the left of x^R_{R+1} on $V_{\pi\gamma'_R}$. (See Fig. 6.) Then, by Lemma 4.6(ii), the two subpaths $\gamma_L[z_{L^*}, w_L]$, $\gamma'_L[z_{L^*}, y_{w_L}]$ are vertex-disjoint other than z_{L^*}, and the two subpaths $\gamma_R[w_R, z_{R^*}]$, $\gamma'_R[y_{w_R}, z_{R^*}]$ are vertex-disjoint other than z_{R^*}.

Case 1: $w_L \neq u$, $w_L \neq v$ and $w_R \neq u$, $w_R \neq v$.

Case 1.1: π_{uv}: $w_0(=u), \ldots, w_L, \ldots, w_R, \ldots, w_l(=v)$. (See Fig. 6(a).) Thus, paths

$$\pi_{z_{L^*} w_L w_R z_{R^*}} = \gamma_L[z_{L^*}, w_L] * \pi_{uv}[w_L, w_R] * \gamma_R[w_R, z_{R^*}],$$

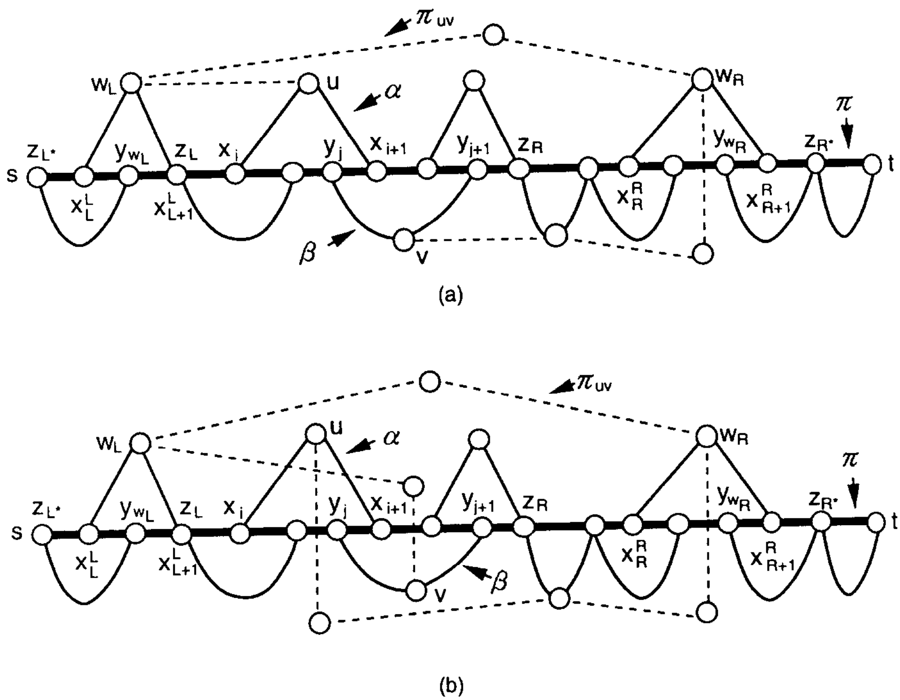

Fig. 6. An illustration of the proof of Lemma 4.9.

$$\pi_{z_{L*}vz_Rz_{R*}} = \gamma'_L[z_{L*}, y_{w_L}] * \pi[y_{w_L}, z_L] * \beta[z_L, z_R] * \pi[z_R, y_{w_R}] * \gamma'_R[y_{w_R}, z_{R*}],$$

$$\pi_{w_Lz_R} = \pi_{uv}[w_L, u] * \alpha[u, z_R],$$

$$\pi_{vw_R} = \pi_{uv}[v, w_R]$$

are internal vertex-disjoint paths. Clearly, the subgraph G^* consisting of the paths is homeomorphic to the graph shown in Fig. 3.

Case 1.2: π_{uv}: $w_0(=u), \ldots, w_R, \ldots, w_L, \ldots, w_l(=v)$. (See Fig. 6(b).) Thus, paths

$$\pi_{z_{L*}w_Lw_Rz_{R*}} = \gamma_L[z_{L*}, w_L] * \pi_{uv}[w_L, w_R] * \gamma_R[w_R, z_{R*}],$$

$$\pi_{z_{L*}z_Lvz_{R*}} = \gamma'_L[z_{L*}, y_{w_L}] * \pi[y_{w_L}, z_L] * \beta[z_L, z_R] * \pi[z_R, y_{w_R}] * \gamma'_R[y_{w_R}, z_{R*}],$$

$$\pi_{z_Lw_R} = \alpha[z_L, u] * \pi_{uv}[u, w_R],$$

$$\pi_{w_Lv} = \pi_{uv}[w_L, v],$$

are internal vertex-disjoint paths. Clearly, the subgraph G^* consisting of the paths is homeomorphic to the graph shown in Fig. 3.

Case 2: One of (1) $w_L = u$ and $w_R \neq v$, (2) $w_L = v$ and $w_R \neq u$, (3) $w_R = u$ and $w_L \neq v$ and (4) $w_R = v$ and $w_L \neq u$. Here, we shall only prove the case (1). Other cases can also be proved by discussions similar to that in the case (1).

(1) $w_L = u$ and $w_R \neq v$ hold. Thus, π_{uv}: $w_l(=w_0 = u)$, $w_R, \dots, w_l(=v)$. This case is considered as a special case of *Case* 1.1, where $z_{L^*} = z_L$, $\gamma_L = \alpha$ and $\gamma'_L = \beta$. From *Case* 1.1, paths

$$\pi_{z_L w_L w_R z_{R^*}} = \alpha_L[z_L, w_L] * \pi_{uv}[w_L, w_R] * \gamma_R[w_R, z_{R^*}],$$

$$\pi_{z_L v z_R z_{R^*}} = \beta[z_L, z_R] * \pi[z_R, y_{w_R}] * \gamma'_R[y_{w_R}, z_{R^*}],$$

$$\pi_{w_L z_R} = \pi_{uv}[w_L, u] * \pi[u, z_R],$$

$$\pi_{v w_R} = \pi_{uv}[v, w_R]$$

are internal vertex-disjoint paths. Clearly, the subgraph G^* consisting of the paths is homeomorphic to the graph shown in Fig. 3.

Case 3: $w_L = u$ and $w_R = v$. This case is also considered as a special case of *Case* 1.

Furthermore, as there are edge-disjoint paths $\alpha[s, z_{L^*}]$, $\beta[s, z_{L^*}]$ and edge-disjoint paths $\alpha[z_{R^*}, t]$, $\beta[z_{L^*}, t]$ in G_π, which have no vertex in G^* other than z_{L^*}, z_{R^*}. Thus, the subgraph obtained by connecting paths $\alpha[s, z_{L^*}]$, $\beta[s, z_{L^*}]$, $\alpha[z_{R^*}, t]$ and $\beta[z_{L^*}, t]$ to G^* is a prohibitive graph by Definition 4.5. □

By Theorem 4.3 and Lemmas 4.5, 4.8, 4.9, the following theorem holds.

Theorem 4.4. *For a probabilistic graph* (G, p), $\underline{\Gamma}_{(G,f,p)} = \Gamma_{(G,p)}$ *iff* G *contains no prohibitive graph as its subgraph.*

5. A method of computing the lower bound

We show a method of computing the lower bound $\underline{\Gamma}_{(G,f,p)}$ for a probabilistic graph (G, p) and an s–t path number f of G. We first wish to recall the procedure **FEDP** and the definition of $\underline{\Gamma}_{(G,f,p)}$ in Section 3.

For a probabilistic graph $(G = (V, E, s, t), p)$ and an s–t path number function f of G, let $\mathscr{U}_{f,\pi_i}$ denote the set of all $U \subseteq E$ for which s–t path π_i is selected as a member of edge-disjoint s–t paths $FEDP(G - U, f)$. Let $p(\mathscr{E}_U)$ be the probability of the event $\mathscr{E}_U$ that all edges of U are failed and all edges of $E - U$ are operative, and $p(\mathscr{E}_{f,\pi_i})$ is the probability of the event that at least one event $\mathscr{E}_U$, for all $U \in \mathscr{U}_{f,\pi_i}$, arises in (G, p). Thus, we have

$$\begin{aligned}
\underline{\Gamma}_{(G,f,p)} &= \sum_{U \subseteq E} |FEDP(G - U, f)| \rho(G - U) \\
&= \sum_{i=1}^{|P_{st}(G)|} \sum_{U \in \mathscr{U}_{f,\pi_i}} \rho(G - U) \\
&= \sum_{i=1}^{|P_{st}(G)|} \sum_{U \in \mathscr{U}_{f,\pi_i}} p(\mathscr{E}_U) \\
&= \sum_{i=1}^{|P_{st}(G)|} p(\mathscr{E}_{f,\pi_i}).
\end{aligned} \tag{5}$$

We can compute the lower bound $\underline{\Gamma}_{(G,f,p)}$ by formula (5) instead of formula (3). To precisely describe the event $\mathscr{E}_{f,\pi_i}$, we need more notations.

For $G=(V,E,s,t)$ and an s–t path number function f of G, let $Q_{f,\pi_i}=\{\pi_j\in P_{st}(G)\mid E(\pi_i)\cap E(\pi_j)\neq\emptyset$ and $f(\pi_j)=j<f(\pi_i)=i\}$. By **FEDP**, for i,j $(1\leqslant j<i\leqslant |P_{st}(G)|)$, we have

$$\mathscr{U}_{f,\pi_i}=\begin{cases}\{U\subseteq E\mid E(\pi_i)\subseteq E-U\}, \\ \quad Q_{f,\pi_i}=\emptyset; \\ \{U\subseteq E\mid E(\pi_i)\subseteq E-U \text{ and } U\notin\mathscr{U}_{f,\pi_j} \text{ for all } \pi_j\in Q_{f,\pi_i}\}, \\ \quad Q_{f,\pi_i}\neq\emptyset.\end{cases}\tag{6}$$

For some events $\mathscr{E}_1,\mathscr{E}_2,\ldots,\mathscr{E}_m$, we denote by $\mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_m$, or $\prod_{i=1}^m\mathscr{E}_i$, for short, the event that all events $\mathscr{E}_1,\mathscr{E}_2,\ldots,\mathscr{E}_m$ arise simultaneously, and by $\mathscr{E}_1+\mathscr{E}_2+\cdots+\mathscr{E}_m$, or $\sum_{i=1}^m\mathscr{E}_i$, for short, the event that at least one event of $\mathscr{E}_1,\mathscr{E}_2,\ldots,\mathscr{E}_m$ arises. The event Ω satisfying $p(\Omega)=1$ is said to be *whole event*, and the event $\emptyset$ satisfying $p(\emptyset)=0$ is said to be *empty event*. Let $\bar{\mathscr{E}}$ denote the *complementary event* of $\mathscr{E}$. Clearly, the following formulas hold for any two events $\mathscr{E}_i,\mathscr{E}_j$.

$$\mathscr{E}_i\overline{\mathscr{E}_i\mathscr{E}_j}=\mathscr{E}_i\bar{\mathscr{E}}_j,\tag{7}$$

$$\mathscr{E}_i\overline{\bar{\mathscr{E}}_i\mathscr{E}_j}=\mathscr{E}_i.\tag{8}$$

For a probabilistic graph $(G=(V,E,s,t),p)$, let $\mathscr{E}_e$ denote the event that edge $e\in E$ is operative (not failed), then $\bar{\mathscr{E}}_e$ denotes the event that edge $e\in E$ is failed. Clearly, probabilities $p(\mathscr{E}_e)$, $p(\bar{\mathscr{E}}_e)$ of events $\mathscr{E}_e$, $\bar{\mathscr{E}}_e$ is $q(e)(=1-p(e))$, $p(e)$, respectively. Moreover, the event $\mathscr{E}_U$, $U\subseteq E$ is as follows:

$$\mathscr{E}_U=\prod_{e\in U}\bar{\mathscr{E}}_e\prod_{e\in E-U}\mathscr{E}_e.\tag{9}$$

For any $E'\subseteq E$, we can easily prove the following:

$$\sum_{U\subseteq E'}\mathscr{E}_U=\sum_{U\subseteq E'}\left(\prod_{e\in U}\bar{\mathscr{E}}_e\prod_{e\in E'-U}\mathscr{E}_e\right)=\Omega.\tag{10}$$

For a probabilistic graph $(G=(V,E,s,t),p)$ and an s–t path number function f of G, when $Q_{f,\pi_i}=\emptyset$, we have

$$\begin{aligned}\mathscr{E}_{f,\pi_i}&=\sum_{U\in\mathscr{U}_{f,\pi_i}}\mathscr{E}_U\\ &=\sum_{U\subseteq E \text{ and } E(\pi_i)\subseteq E-U}\mathscr{E}_U \quad \text{(by (6))}\\ &=\prod_{e\in E(\pi_i)}\mathscr{E}_e\sum_{U'\subseteq E-E(\pi_i)}\mathscr{E}_{U'} \quad \text{(by (9))}\\ &=\prod_{e\in E(\pi_i)}\mathscr{E}_e \quad \text{(by (10))}.\end{aligned}\tag{11}$$

Clearly, $p(\mathscr{E}_{f,\pi_i})(=\prod_{e\in E(\pi_i)} q(e))$ is efficiently computed. However, when $Q_{f,\pi_i} \neq \emptyset$,

$$\begin{aligned}
\mathscr{E}_{f,\pi_i} &= \sum_{U\in \mathscr{U}_{f,\pi_i}} \mathscr{E}_U \\
&= \sum_{U\subseteq E \text{ and } E(\pi_i)\subseteq E-U \text{ and } U\notin \mathscr{U}_{f,\pi_j} \text{ for all } \pi_j\in Q_{f,\pi_i}} \mathscr{E}_U \quad \text{(by (6))} \\
&= \left(\sum_{U\subseteq E \text{ and } E(\pi_i)\subseteq E-U} \mathscr{E}_U\right)\left(\sum_{U\subseteq E \text{ and } U\notin \mathscr{U}_{f,\pi_j} \text{ for all } \pi_j\in Q_{f,\pi_i}} \mathscr{E}_U\right) \\
&= \left(\prod_{e\in E(\pi_i)} \mathscr{E}_e \sum_{U'\subseteq E-E(\pi_i)} \mathscr{E}_{U'}\right)\left(\prod_{\pi_j\in Q_{f,\pi_i}} \sum_{U\notin \mathscr{U}_{f,\pi_j} \text{ and } U\subseteq E} \mathscr{E}_U\right) \quad \text{(by (9))} \\
&= \prod_{e\in E(\pi_i)} \mathscr{E}_e \prod_{\pi_j\in Q_{f,\pi_i}} \bar{\mathscr{E}}_{f,\pi_j} \quad \text{(by (10) and the definition of } \mathscr{E}_{f,\pi_i}) \qquad (12)
\end{aligned}$$

is obtained. Moreover, by (7), (8), each of events $\bar{\mathscr{E}}_{f,\pi_j}$, for all $\pi_j \in Q_{f,\pi_i}$, can be abbreviated to the event $\bar{\mathscr{E}}_{f,\pi_j/\pi_i}$ in which both $\mathscr{E}_e$ and $\bar{\mathscr{E}}_e$, for any $e\in E(\pi_i)$, does not appear. $\bar{\mathscr{E}}_{Q_{f,\pi_i}/\pi_i} = \prod_{\pi_j\in Q_{f,\pi_i}} \bar{\mathscr{E}}_{f,\pi_j/\pi_i}$ and $\prod_{e\in E(\pi_i)} \mathscr{E}_e$ are independent. Hence, for π_i satisfying $Q_{f,\pi_i} \neq \emptyset$, we have

$$p(\mathscr{E}_{f,\pi_i}) = \left(\prod_{e\in E(\pi_i)} q(e)\right) p(\bar{\mathscr{E}}_{Q_{f,\pi_i}/\pi_i}).$$

The lower bound $\Gamma_{(G,f,p)}$ does not seem to be efficiently computed in general, as the events $\mathscr{E}_{f,\pi_j/\pi_i}$, for all $\pi_j \in Q_{f,\pi_i}$, are not necessarily independent of each other, i.e., the probability $p(\mathscr{E}_{f,\pi_i})$ does not seem to be efficiently computed in general, and in addition to this, $|P_{st}(G)|$ is not polynomial in the size of a general graph G.

6. Concluding remarks

For a probabilistic graph, we proposed a lower bound for estimating the expected maximum number of edge-disjoint s–t paths. The necessary and sufficient conditions with respect to both s–t path number function and graph construction, where this lower bound coincides with the expected maximum number of edge-disjoint s–t paths, are clarified. A method of computing this lower bound is also given, although by this computing method the lower bound does not seem to be efficiently computed for a general probabilistic graph.

However, for a probabilistic one-layered s–t graph, (a two-terminal graph where the subgraph obtained by deleting its s, t is exactly a simple path. Fig. 7 illustrates an example of one-layered s–t graph.) as it satisfies the necessary and sufficient conditions and the number of all its s–t paths is a polynomial function in the number of its vertices, the lower bound based on its exact s–t path number function can efficiently

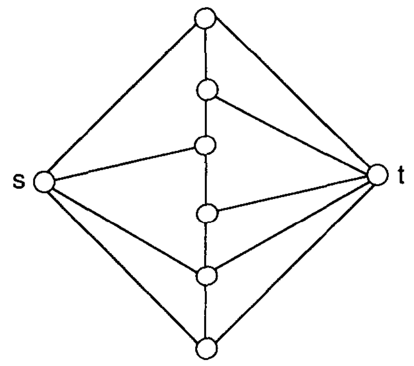

Fig. 7. A one-layered s–t graph.

be computed by the computing method shown in Section 5, i.e., the expected maximum number of edge-disjoint s–t paths in a probabilistic one-layered s–t graph can efficiently be computed. Detailed descriptions of these proofs are lengthy and to be reported elsewhere.

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